

# ON GLOBAL NON-OSCILLATION OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH POLYNOMIAL COEFFICIENTS

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**ABSTRACT.** In this note we show that a linear ordinary differential equation with polynomial coefficients is globally non-oscillating in  $\mathbb{CP}^1$  if and only if it is Fuchsian, and at every its singular point any two distinct characteristic exponents have distinct real parts. As a byproduct of our study, we obtain a new explicit upper bound for the number of zeros of exponential polynomials in a horizontal strip.

## 1. INTRODUCTION

Let us recall the classical notions of disconjugacy and non-oscillation of a linear ordinary differential equation, see e.g. [3].

**Definition 1.** A linear ordinary differential equation of order  $k$

$$a_k(z)y^{(k)} + a_{k-1}(z)y^{(k-1)} + \dots + a_0(z)y = 0, \quad (1)$$

with continuous coefficients  $a_j(z)$ ,  $j = 0, \dots, k$  defined in a neighborhood of some simply-connected subset  $I$  of  $\mathbb{R}$  or  $\mathbb{C}$ , is called *disconjugate* (resp. *non-oscillating*) in  $I$ , if every its nontrivial solution has in  $I$  at most  $k - 1$  zeros (resp. finitely many zeros) counted with multiplicities.

Observe that every equation (1) is disconjugate in any sufficiently small interval in  $\mathbb{R}$  (resp. any sufficiently small disk in  $\mathbb{C}$ ) centered at an arbitrary point  $z_0 \in \mathbb{R}$  (resp.  $z_0 \in \mathbb{C}$ ) such that  $a_k(z_0) \neq 0$ . Analogously, every equation (1) is non-oscillating in any compact simply-connected set free from the roots of  $a_k(z)$ .

The study of different aspects and criteria of disconjugacy and non-oscillation has been an active topic in the past. While there exist satisfactory criteria of disconjugacy for the second order equations, the situation with the higher order equations is more complicated. A number of necessary/sufficient conditions of disconjugacy for subsets of  $\mathbb{R}$  and  $\mathbb{C}$  are known in the literature mostly dating back at least four decades, see e.g. [10], [8], [9]. In the case of equations of order 2, disconjugacy is closely related to Sturm separation theorems; for higher order equations there is a related version of multiplicative Sturmian theory developed in [12].

In this paper, for a linear differential equation with polynomial coefficients, we introduce the notion of its global non-oscillation in  $\mathbb{CP}^1$  by which we mean its classical non-oscillation in an arbitrary *open contractible* domain obtained after the removal from  $\mathbb{CP}^1$  of an appropriate cut connecting all the singular points. Although oscillation/non-oscillation in the complex domain have been studied since the 1920's, (see e.g. [5]), the notion of global non-oscillation seems to be new. As

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an experienced reader can easily guess, the main motivation for our consideration comes from the second part of Hilbert's 16th problem.

Consider a linear homogeneous differential equation

$$P_k(z)y^{(k)} + P_{k-1}(z)y^{(k-1)} + \dots + P_0(z)y = 0, \quad (2)$$

with polynomial coefficients  $P_k(z), P_{k-1}(z), \dots, P_0(z)$ , and  $GCD(P_k, P_{k-1}, \dots, P_0) = 1$ . Let  $S$  be the set of all singular points of (2) in  $\mathbb{CP}^1$ , i.e., the set of all roots of  $P_k(z)$  (together with  $\infty$  if some of the limits  $\lim_{z \rightarrow \infty} z^j P_{k-j}(z)/P_k(z)$ ,  $j = 0, \dots, k$  is infinite). For a given equation (2), let  $d$  denote the cardinality of  $S$ .

**Definition 2.** A system  $\overline{\mathcal{C}} := \{\mathcal{C}_j\}_{j=1}^{d-1}$  of smooth Jordan curves in  $\mathbb{CP}^1$ , each of them connecting a pair of distinct singular points, is called *an admissible cut* for equation (2) if and only if: a) for any  $i \neq j$ , the intersection  $\mathcal{C}_i \cap \mathcal{C}_j$  is either empty or consists of their common endpoint; b) the union  $\cup_{j=1}^{d-1} \mathcal{C}_j$  is topologically a tree in  $\mathbb{CP}^1$ , i.e., the complement  $\mathbb{CP}^1 \setminus \cup_j \mathcal{C}_j$  is contractible; c) each  $\mathcal{C}_j$  has a well-defined tangent vector at each of its two endpoints.

In particular, there exist admissible cuts consisting of straight segments connecting the singular points of (2).

**Definition 3.** Equation (2) is called *globally non-oscillating* if, for any its admissible cut  $\overline{\mathcal{C}}$ , every its nontrivial solution has finitely many zeros in  $\mathbb{C} \setminus \overline{\mathcal{C}}$ .

The main result of this paper is the following criterion of global non-oscillation.

**Theorem 4.** *Equation (2) is globally non-oscillating if and only if:*

- (i) *it is Fuchsian;*
- (ii) *at each singular point all distinct characteristic exponents have pairwise distinct real parts.*

**Remark 5.** One can easily notice that (2) is globally non-oscillating if and only if some (and therefore any) domain  $\mathbb{CP}^1 \setminus \overline{\mathcal{C}}$  can be covered by finitely many open disconjugacy domains. Observe that if one knows such a covering, then one gets an immediate upper bound for the total number of zeros of nontrivial solutions of (2) in  $\mathbb{CP}^1 \setminus \overline{\mathcal{C}}$ . Namely, if the number of open disconjugacy domains covering  $\mathbb{CP}^1 \setminus \overline{\mathcal{C}}$  equals  $l$ , then any nontrivial solution of (2) has there at most  $(k-1)l$  zeros counted with multiplicities.

In view of Remark 6 the following problem is of fundamental importance.

**Main Problem.** Given an arbitrary equation (2) satisfying the assumptions of Theorem 4, estimate from above the number of disconjugacy domains which can form an open covering of  $\mathbb{CP}^1 \setminus \overline{\mathcal{C}}$ , for some admissible cut  $\mathcal{C}$ .

Observe that in case of a Schrödinger equation

$$-y'' + P(z)y = 0$$

with a polynomial potential  $P(z)$ , there is a classical construction of such coverings using the Schwarzian derivative of two linearly independent solutions of the latter equation which goes back to R. Nevanlinna, [11].

**Remark 6.** Let us also mention that Proposition 9 below, which is an important technical tool used to prove Theorem 4, is a new result in the classical area of the upper bounds for the number of zeros of exponential polynomials and, therefore, it is of independent interest. Such upper bounds are required in a wide range of mathematical disciplines, from applied mathematics to number theory. Essential progress in this area has been made in the 70's in the papers [13], [15], [16]. But, to

the best of our knowledge, in all the previous literature one only considered compact subdomains in  $\mathbb{C}$ , mainly disks and rectangles, while Proposition 9 considers the case of an infinite strip.

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## 2. PROOFS

Our proof of Theorem 4 consists of several steps.

Step 1. The necessity of Conditions (i) and (ii) for global non-oscillation of equation (2).

Indeed, if (2) has a non-Fuchsian singularity at  $p \in \mathbb{CP}^1$ , then, for any sufficiently small  $\epsilon > 0$ , almost any solution of (2) has infinitely many zeros in the  $\epsilon$ -neighborhood of  $p$  with a removed straight segment connecting  $p$  with some point on the bounding circle. This property contradicts to global non-oscillation. To finish Step 1, consider a Fuchsian singularity of (2) with two distinct characteristic exponents of the form  $a + b_1 I$  and  $a + b_2 I$ . Then there exists a solution of (2) with the leading term  $z^{a + \frac{(b_1 + b_2)I}{2}} \cos\left(\frac{b_1 - b_2}{2} \ln z\right)$ . Such a solution has infinitely many zeros accumulating to  $p$  which are located close to the horizontal line passing through  $p$ . This again contradicts to global non-oscillation.

Step 2. Reduction to small neighborhoods of singular points.

For any sufficiently small  $\epsilon > 0$ , construct a simply-connected domain  $U_\epsilon \subset \mathbb{CP}^1$  by: a) taking the large disk  $\{|z| < \epsilon^{-1}\}$  with the  $\epsilon$ -neighborhoods of all zeros of  $P_k$  removed, b) making cuts by straight segments between the bounding circles so that the obtained domain becomes contractible.

The following complex analogue of the classical de la Vallée Poussin theorem [4] is proved in [17, Theorem 2.6, Corollary 2.7].

**Lemma 7.** *Consider a homogeneous monic linear ordinary differential equation with holomorphic coefficients*

$$y^{(k)} + a_{k-1}(t)y^{(k-1)} + \dots + a_0(t)y = 0, \quad t \in \mathbb{C}.$$

*Then the variation of the argument of any solution  $y(t)$  along a circular arc  $\gamma$  of a known length is explicitly bounded in terms of the uniform upper bounds  $A_i = \sup_{t \in \gamma} |a_i(t)|$ ,  $i = 0, \dots, k-1$ .*

Lemma 7 implies an explicit upper bound  $B(\epsilon)$  for the number of zeros of any solution of (2) in  $U_\epsilon$ . More exactly, the upper bound will depend on the upper bounds on the restrictions of  $a_j$  to  $\partial U_\epsilon$ . The latter are polynomial in  $\epsilon^{-1}$  if the coefficients  $a_j$  are polynomials, so the upper bound is also polynomial in  $\epsilon^{-1}$ .

**Remark 8.** Observe that, for any admissible system of cuts  $\overline{\mathcal{C}}$  and any sufficiently small  $\epsilon$ , the domain  $\mathbb{CP}^1 \setminus \overline{\mathcal{C}}$  can be covered by finitely many  $U_\epsilon$  (choosing different straight lines connecting the bounding circles) and finitely many sectors of finite radii centered at the singular points of (2). This observation reduces the proof of Theorem 4 to providing finite upper bounds for the number of zeros of solutions of (2) in these sectors, see below.

Step 3. Equations with constant coefficients. ("Reduction" to the case of equations with constant coefficients in a neighborhood of a Fuchsian singularity is obtained by using the logarithmic chart centered at the singularity. See also Steps 4-5.)

**Proposition 9.** *For any  $\alpha \geq 0$  and for any equation*

$$EQ: \quad a_k y^{(k)} + a_{k-1} y^{(k-1)} + \dots + a_0 y = 0, \quad a_j \in \mathbb{C}, \quad a_k \neq 0 \quad (3)$$

*such that all its distinct characteristic roots have distinct real parts,*

- (1) *there exists an upper bound  $\sharp(EQ, \alpha)$  for the number of zeros of all nontrivial solutions of (3) in the horizontal strip  $\{\Pi_\alpha : |\Im(z)| \leq \alpha\}$ . (Here zeros are counted with multiplicities.)*
- (2) *in the generic case when all roots  $\lambda_j$ ,  $j = 1, \dots, k$  of the characteristic equation of (3) are simple, with  $\Re \lambda_1 < \Re \lambda_2 < \dots < \Re \lambda_k$ , we get*

$$\sharp(EQ, \alpha) \leq (k-1)^2 + \frac{2}{\pi}(k-1)\mathcal{L}(EQ) [\alpha(\Xi + 2) + \Theta \log 4], \quad (4)$$

*where  $\mathcal{L}(EQ)$  is the length of the shortest polygonal path passing through all  $\lambda_j$  and*

$$\Theta := \max_{1 \leq j \leq k-1} |\Re(\lambda_j) - \Re(\lambda_{j+1})|^{-1}, \quad \Xi := \max_{1 \leq j \leq k-1} \left| \frac{\Im \lambda_j - \Im \lambda_{j+1}}{\Re \lambda_j - \Re \lambda_{j+1}} \right|.$$

The case of multiple characteristic roots will be considered in Step 4.

Our approach to the proof of Proposition 9 is inspired by the Wiman-Valiron theory, see [14]. The main construction below has a strong resemblance with the notion of a tropical polynomial in the modern tropical geometry. The proof itself is based on a rather long sequence of lemmas and the last argument is given at the end of Step 5.

The general solution of (3) is given by:

$$y = \sum_j A_j(z) e^{\lambda_j z}, \quad \text{where } \deg A_j(z) = n_j, \quad \sum (n_j + 1) = k. \quad (5)$$

Define the domain of a single term  $y$ -dominance in  $\Pi_\alpha$  as

$$G(y, \alpha) := \{z \in \Pi_\alpha \mid \exists j = j(z), \exists \epsilon > 0 : |A_j(z) e^{\lambda_j z}| \geq (1-\epsilon) \sum_{i \neq j} |A_i(z) e^{\lambda_i z}|\}. \quad (6)$$

Note that  $G(y, \alpha)$  may contain at most  $\min n_j \leq k$  zeros of  $y$ , namely the common zeros of all  $A_j(z)$ . In particular,  $G(y, \alpha)$  contains no zeros of  $y$  at all in the case of simple characteristic exponents.

**Lemma 10.** *The complement  $\Pi_\alpha \setminus G(y, \alpha)$  can be covered by at most  $k + k^2 + k^3$  horizontal boxes (of height  $2\alpha$ ) of the total width not exceeding*

$$k^2(k+1)(4\Theta \ln k + 4\alpha\Xi + 4\alpha) + 8k^2\Theta.$$

We first consider the case of simple characteristic exponents  $\lambda_j$ . This case is more transparent and the resulting estimates seem to be of correct order of magnitude. In this case the polynomials  $A_j(z)$  are constants and will be denoted by  $a_j$ .

**Lemma 11.** *In the case of simple characteristic exponents  $\lambda_j$ , the complement  $\Pi_\alpha \setminus G(y, \alpha)$  can be covered by at most  $k-1$  horizontal boxes (of height  $2\alpha$ ) of the total width not exceeding*

$$2\alpha(k-1)\Xi + 2(k-1)\Theta \ln 4. \quad (7)$$

The principal case in Lemma 11 is  $\alpha = 0$ , i.e.  $\Pi_0 = \mathbb{R}$ .

**Lemma 12.** *In the above notations,  $\mathbb{R} \setminus G(y, 0)$  is contained in the union of at most  $k-1$  closed intervals of the total length less than or equal to  $2(k-1) \ln 4 \cdot \Theta$ .*

To prove Lemma 12, we need an additional statement. In  $\mathbb{R}^2$  with coordinates  $(\mu, \phi)$  consider the 1-parameter family  $\{Pt_j(u)\}_{j=1}^k$  of  $k$  points given by

$$\mu = \Re(\lambda_j), \quad \phi = \ln |a_j e^{\lambda_j u}|,$$

where  $u$  is a real-valued parameter. For a given value of  $u \in \mathbb{R}$ , introduce the piecewise-linear function  $\phi_u(\mu)$  as the *least concave majorant* of  $\{Pt_j(u)\}_{j=1}^k$ . By this we mean the minimal concave function  $\phi_u(\mu)$  defined in the interval  $[\Re(\lambda_1), \Re(\lambda_k)]$  such that all points  $\{Pt_j(u)\}_{j=1}^k$  lie non-strictly below its graph, i.e. have their  $\phi$ -coordinate smaller than or equal to that of  $\phi_u(\mu)$ . (One can easily see that the graph of  $\phi_u(\mu)$  is the upper part of the boundary of the convex hull of  $\{Pt_j(u)\}_{j=1}^k$  connecting  $Pt_1(u)$  and  $Pt_k(u)$ .) Observe that, for any  $u \in \mathbb{R}$ ,

$$\phi_u(\mu) = \phi_0(\mu) + u\mu. \quad (8)$$

**Lemma 13.** *If, for  $j = 1, \dots, k-1$ ,*

$$|\phi_u(\Re(\lambda_{j+1})) - \phi_u(\Re(\lambda_j))| \geq \ln 4,$$

*then  $u \in G(y, 0)$ .*

*Proof of Lemma 13.* Define the central index of  $\phi_u(\mu)$  by the formula:

$$i(u) := \{i \mid \Re(\lambda_i) \text{ is the point of the global maximum for } \phi_u(\mu)\},$$

comp. Ch. 1, [14]. Then for any  $j \neq i$ ,

$$|a_j e^{\lambda_j u}| \leq \exp(\phi_u(\Re(\lambda_j))) \leq 4^{-|j-i(u)|} |a_{i(u)} e^{\lambda_{i(u)} u}|.$$

Therefore the inequality in the definition (6) of  $G(y, 0)$  follows after the summation of a geometric series.  $\square$

**Corollary 14.** *If  $-u$  lies outside the  $\ln 4 \cdot \Theta$ -neighborhood of the set of all slopes of  $\phi_0(\mu)$ , then  $u \in G(y, 0)$ .*

*Proof.* Formula (8) implies that each slope of  $\phi_u(\mu)$  equals the sum of the respective slope of  $\phi_0(\mu)$  and  $u$ . Therefore in the considered case, the absolute values of all slopes of  $\phi_u(\mu)$  exceed  $\ln 4 \cdot \Theta$ , and the statement follows immediately from Lemma 13.  $\square$

*Proof of Lemma 12.* The  $\ln 4 \cdot \Theta$ -neighborhood of the set of slopes of  $\phi_0(\mu)$  consists of the union of at most  $k-1$  intervals of total length not exceeding  $2(k-1) \ln 4 \cdot \Theta$ .  $\square$

*Proof of Lemma 11.* Consider the general case of Lemma 11 with  $\alpha \geq 0$ .

We repeat the above construction of Lemma 12 for  $z$  running along the horizontal line  $\Im z = v$  with  $|v| \leq \alpha$ . For every fixed  $v$ , consider in  $\mathbb{R}^2$  with coordinates  $(\mu, \phi)$ , the 1-parameter family  $\{Pt_j^v(u)\}_{j=1}^k$  of  $k$  points given by

$$\mu = \Re(\lambda_j), \quad \phi = \ln |a_j e^{\lambda_j(u+Iv)}|$$

where  $u$  is a real parameter. Introduce  $\phi_u^v(\mu)$  as the *least concave majorant* of  $\{Pt_j^v(u)\}_{j=1}^k$ , for a given value of  $u \in \mathbb{R}$ . Observe that, for any  $u \in \mathbb{R}$ ,

$$\phi_u^v(\mu) = \phi_0^v(\mu) + u\mu. \quad (9)$$

Now consider the set  $Sl_\alpha := \cup_{-\alpha \leq v \leq \alpha} \{k_j(v)\}$ , where  $k_j(v)$  are the slopes of  $\phi_0^v(\mu)$ . We claim that  $Sl_\alpha$  is the union of at most  $k-1$  closed intervals. Indeed, the set of slopes  $\{k_j(v)\}$  changes continuously with  $v$ , and consists of no more than  $k-1$  points for each fixed  $v$ .

Moreover, as  $\ln |a_j e^{I\lambda_j v}| = \ln |a_j| - v \Im \lambda_j$ , the points  $\{Pt_j^v(0)\}_{j=1}^k$  defining  $\phi_0^v(\mu)$  depend linearly on  $v$ , namely they move up or down as  $v$  changes. The inequality

$$\left| \frac{\partial k_j(v)}{\partial v} \right| \leq \Xi$$

is straightforward. Therefore the total length of  $Sl_\alpha$  is at most  $2\alpha\Xi(k-1)$ .

By Corollary 14, if  $-u$  lies outside the  $\ln 4 \cdot \Theta$ -neighborhood of  $Sl_\alpha$ , then, for any  $|v| \leq \alpha$ ,  $u + Iv$  lies in  $G(y, \alpha)$  which settles Lemma 11.  $\square$

Step 4. Case of multiple characteristic exponents.

In this case the dependence on  $v$  of (analogs of) points  $Pt_j^v(u)$  seems to be more complicated, and we are forced to consider the slopes of *all* chords connecting these points, and not only those which lie on the boundary of their convex hull. This apparently leads to an excessive upper bound of the total width of  $\Pi_\alpha \setminus G(y, \alpha)$ .

*Proof of Lemma 10.* Consider the absolute value  $r_{jj'}$  of the ratio of any two terms in (5). The complement  $\Pi_\alpha \setminus G(y, \alpha)$  lies in the union  $\Sigma$  of the sets  $\Sigma_{jj'}^o = \{|\ln r_{jj'}(z)| \leq \ln k\}$ , where  $r_{jj'}$  is the absolute value of the ratio of two terms in (5).

We can write

$$\ln r_{jj'} = \ln |A_j/A_{j'}| - v\xi_{jj'}\theta_{jj'} + \theta_{jj'}u, \quad (10)$$

where

$$\theta_{jj'} = \Re(\lambda_j - \lambda_{j'}), \quad \xi_{jj'} = \theta_{jj'}^{-1} \Im(\lambda_j - \lambda_{j'}).$$

Set  $W = \{|\Re(z - z_i)| \geq 4k\Theta\}$ , where  $z_i$  runs over all roots of all  $A_j$ . Outside  $W$  we have

$$\left| \frac{\partial}{\partial u} \ln |A_j/A_{j'}| \right|, \left| \frac{\partial}{\partial v} \ln |A_j/A_{j'}| \right| \leq \frac{|\theta_{jj'}|}{2}.$$

Additionally,

$$\Sigma_{jj'}^o \subset \Sigma_{jj'} = \{u + Iv \in \Pi_\alpha : |\ln r_{jj'}(u)| \leq \ln k + \alpha|\xi_{jj'}\theta_{jj'}| + \alpha|\theta_{jj'}|\},$$

outside  $W$ . Note that  $\Sigma_{jj'}$  is the union of boxes, since its definition is independent of  $v$ .

Therefore,

$$\left| \frac{\partial \ln r_{jj'}}{\partial u} \right| \geq \frac{|\theta_{jj'}|}{2}, \quad (11)$$

outside  $W$ .

Thus  $\Sigma_{jj'}$  intersects each connected component of  $\mathbb{R} \setminus W$  in an interval of length at most  $4|\theta_{jj'}|^{-1} \ln k + 4\alpha|\xi_{jj'}| + 4\alpha$ . In other words,  $\Sigma_{jj'} \setminus W$  is the union of at most  $k+1$  boxes of total width not exceeding  $(k+1)(4 \ln k |\theta_{jj'}|^{-1} + 4\alpha|\xi_{jj'}| + 4\alpha)$ .

Taking the union over all possible pairs  $(j, j')$ , we conclude that  $\Sigma \setminus W$  lies in the union of at most  $k^2(k+1)$  boxes of total width at most  $k^2(k+1)(4\Theta \ln k + 4\alpha\Xi + 4\alpha)$ . As  $W \cap \Pi_\alpha$  is the union of at most  $k$  boxes of width at most  $8k\Theta$  each, we obtain that  $\Sigma$  lies in the union of at most  $k + k^2 + k^3$  boxes of total width at most  $k^2(k+1)(4\Theta \ln k + 4\alpha\Xi + 4\alpha) + 8k^2\Theta$ .  $\square$

Finally let us explain how Lemmas 10 and 11 imply Proposition 9. Consider the space  $QP_\Lambda = \{\sum_j A_j(z)e^{\lambda_j z}, A_j \in \mathbb{C}[z]\}$  of dimension  $k = \sum (1 + \deg A_j)$  consisting of exponential polynomials, where  $\Lambda = \{\lambda_j\} \subset \mathbb{C}$  is some finite set. The following result was proven in [6].

**Theorem 15** ([6]). *The number of zeros of any function  $f \in QP_\Lambda$  in a bounded convex domain  $U$  does not exceed*

$$k - 1 + \frac{1}{\pi} \mathcal{L}(\Lambda) \text{diam}(U), \quad (12)$$

where  $\mathcal{L}(\Lambda)$  is the length of a shortest polygonal path passing through all points of  $\Lambda$ .

Theorem 15 immediately implies an estimate on the number of zeros of  $y$  in the boxes  $B_j$  of Lemma 10 and 11. In the case of simple characteristic exponents (second part of Proposition 9)

$$\sum \text{diam } B_j \leq 2(k-1) [\alpha\Xi + \Theta \log 4] + 4(k-1)\alpha,$$

and (4) follows.

Step 5. Equation with non-constant coefficients in a semistrip.

In general, solutions of (2) considered in the logarithmic chart near its Fuchsian singularity have the form

$$y = \sum_j \tilde{A}_j(z) e^{\lambda_j z}, \quad (13)$$

where

$$\tilde{A}_j(z) = \sum_{r=0}^{n_j} a_{j,r} z^{n_j-r} (1 + \epsilon_{j,r}),$$

and  $\epsilon_{j,r}$  is  $2\pi I$ -periodic,  $\epsilon_{j,r} = O(e^z)$  in any semistrip  $\Pi_{\alpha,\beta} = \{|\Im z| \leq \alpha, \Re z \leq \beta\}$  for some  $\beta$  depending on (2) only. To simplify our notation, let us assume that  $\beta < 0$ .

**Lemma 16.** *Assume that  $|\epsilon_{j,r}| < C e^{\Re z}$  in  $\Pi_{\alpha,\beta}$ . Let  $A_j(z) = \sum_{r=0}^{n_j} a_{j,r} z^{n_j-r}$  and  $W$  be as in the proof of Lemma 10. Then*

$$|\log |\tilde{A}_j/A_j|| \leq C_{EQ,\alpha} \text{ in } \Pi_{\alpha,\beta} \setminus W, \quad (14)$$

where  $C_{EQ,\alpha}$  is some constant depending on  $\alpha$  and (2) only.

*Proof.* Let  $\mathring{A}(z) = \sum |a_{j,r}| |z|^r$ . Evidently,  $\mathring{A}_j(z) \leq \prod_m (|z| + |z_m|)$ , where  $z_m$  are the roots of  $A_j(z)$ . Also,  $|\tilde{A}_j - A_j| \leq C e^{\Re z} \mathring{A}_j(z)$ . Therefore, for  $z \in \Pi_{\alpha,\beta} \setminus W$ , we get

$$\begin{aligned} |\log |\tilde{A}_j/A_j|| &\leq C e^{\Re z} \frac{\mathring{A}_j(z)}{|A_j(z)|} \leq C e^{\Re z} \prod \frac{|z| + |z_m|}{||z| - |z_m||} \leq \\ &\leq C e^{\Re z} |z|^\ell \prod_{|z_m| < 2|z|} \frac{1 + |z_m|/|z|}{||z| - |z_m||} \prod_{|z_m| > 2|z|} \frac{|z/z_m| + 1}{1 - |z/z_m|} \leq C e^{\Re z} |z|^\ell \left(\frac{3}{4k\Theta}\right)^\ell 3^{k-\ell}. \end{aligned} \quad (15)$$

Clearly, the latter function can be majorized by some number depending on  $C, \alpha$  and  $k, \Theta$  only. The constants  $C, k, \Theta$  are determined by (2).  $\square$

**Remark 17.** *Actually, dependence of  $C_{EQ,\alpha}$  on  $\alpha$  is very simple (as  $O(\alpha^k)$  as  $\alpha \rightarrow \infty$ ), but we do not need this.*

**Lemma 18.** *In the above notation, the zeros of  $y$  in  $\Pi_{\alpha,\beta}$  lie in at most  $k + k^2 + k^3$  boxes of total width at most*

$$k^2(k+1)(4\Theta \ln k + 4\alpha\Xi + 4\alpha + 4C_{EQ,\alpha}) + 8k^2\Theta.$$

*Proof.* We repeat the proof of Lemma 10. Namely, consider the absolute value  $\tilde{r}_{jj'}$  of the ratio of any two terms in (13). The complement  $\Pi_\alpha \setminus G(y, \alpha)$  lies in the union  $\Sigma$  of the sets  $\tilde{\Sigma}_{jj'}^\circ = \{|\ln \tilde{r}_{jj'}(z)| \leq \ln k\}$ . But, according to Lemma 16,  $|\log \tilde{r}_{jj'} - \log r_{jj'}| \leq C_{QE,\alpha}$ , where  $r_{jj'}$  was defined in the proof of Lemma 10. So, it is enough to require  $|\ln \tilde{r}_{jj'}(z)| \leq \ln k + C_{EQ,\alpha}$ , i.e. outside  $W$

$$\tilde{\Sigma}_{jj'}^\circ \subset \tilde{\Sigma}_{jj'} = \{u + Iv \in \Pi_\alpha : |\ln r_{jj'}(u)| \leq \ln k + \alpha|\xi_{jj'}\theta_{jj'}| + \alpha|\theta_{jj'}| + C_{EQ,\alpha}\}.$$

Repeating the same arguments as in Lemma 10 with  $\tilde{\Sigma}_{jj'}$  instead of  $\Sigma_{jj'}$ , we arrive at the required estimates.  $\square$

*Proof of Theorem 4.* Let  $y^{(k)} + b_1(z)y^{(k-1)} + \dots + b_k y = 0$  be the reduced form (=divided by its leading term) of (2) in the logarithmic chart near its Fuchsian singularity. Assume that  $b_j(z)$  are bounded by  $C$  in  $\Pi_{\alpha,\beta}$  (The Fuchsian property implies that each  $b_j(z)$  tends to some finite limit when  $z \rightarrow \infty$  in  $\Pi_{\alpha,\beta}$ ).

Example in [17] immediately following after Corollary 2.7 of this paper, implies that  $y$  has at most  $2(k+1) + \frac{k+1}{\log(9/4)} \ell C$  zeros in  $\Pi_{\alpha,\beta}$ , where

$$\ell \leq 2k^2(k+1)(4\Theta \ln k + 4\alpha\Xi + 4\alpha + 4C_{EQ,\alpha}) + 16k^2\Theta + 4(k+k^2+k^3)\alpha,$$

is the total perimeter of all boxes appearing in Lemma 18.

After going back from logarithmic chart to the original coordinate, we obtain an upper bound for the number of zeros of any solution of (2) in the sector  $\{|z-p| \leq e^\beta, |\arg z| \leq \alpha\}$  at the Fuchsian singular point  $p$ . □

The sequence of steps 1-5 settles Theorem 4.

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